National
Qualifications
2023

2023 Mathematics

Advanced Higher - Paper 1

Finalised Marking Instructions

Marking Instructions for each question

Question			Generic scheme	Illustrative scheme	Max mark
1.			•¹ evidence of use of product rule with one term correct [1,2] •² complete differentiation	•¹ $7\tan 2x + 7x(...)$ **OR** $(...)\tan 2x + 7x \times 2\sec^2 2x$ •² $7\tan 2x + 14x\sec^2 2x$	2

Notes:

1. For a candidate who produces one term only, award 0/2.
2. Where a candidate equates $\frac{dy}{dx}$ to y, •¹ is not available.

Commonly Observed Responses:

Question			Generic scheme	Illustrative scheme	Max mark
2.			•[1] write template [1]	•[1] $\frac{3x^2-x-14}{(x+3)(x-1)^2}=\frac{A}{x+3}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$	3
			•[2] form equation and find one constant	•[2] $3x^2-x-14=A(x-1)^2+B(x+3)(x-1)+C(x+3)$ and $A=1$ or $B=2$ or $C=-3$	
			•[3] find remaining constants and substitute [2]	•[3] $\frac{1}{x+3}+\frac{2}{x-1}-\frac{3}{(x-1)^2}$	

Notes:

1. Award 0/3 if an incorrect template has been used.
2. Do not accept $+-$ at •[3]

Commonly Observed Response:

•[1] $\frac{3x^2-x-14}{(x+3)(x-1)^2}=\frac{A}{x+3}+\frac{Bx+C}{(x-1)^2}$

•[2] $3x^2-x-14=A(x-1)^2+(Bx+C)(x+3)$

and

$A=1$, $B=2$ and $C=-5$

•[3] $\frac{1}{x+3}+\frac{2}{x-1}-\frac{3}{(x-1)^2}$

Question			Generic scheme	Illustrative scheme	Max mark
3.			•[1] set up augmented matrix [1]	•[1] $\left(\begin{array}{ccc\|c} 1 & -3 & 1 & -1 \\ 3 & -2 & 4 & 11 \\ 1 & 4 & 2 & 15 \end{array}\right)$	3
			•[2] obtain two zeros	•[2] $\left(\begin{array}{ccc\|c} 1 & -3 & 1 & -1 \\ 0 & 7 & 1 & 14 \\ 0 & 7 & 1 & 16 \end{array}\right)$	
			•[3] write down conclusion with justification [2]	•[3] eg $\left(\begin{array}{ccc\|c} 1 & -3 & 1 & -1 \\ 0 & 7 & 1 & 14 \\ 0 & 0 & 0 & 2 \end{array}\right)$ (or statement relating to $14 \neq 16$ at •[2]) so inconsistent	

Notes:

1. Where a candidate equates a 3×3 matrix to a 3×1 matrix, •[1] is not available. Otherwise, accept eg $x, y, z, =$ left in.
2. For •[3], candidates who arrive at an augmented matrix which produces a unique solution, or infinitely many solutions, there is no requirement to determine solutions.

Commonly Observed Responses:

Question			Generic scheme	Illustrative scheme	Max mark
4.			•¹ integrate to find "$uv -$" [1,2]	•¹ $\frac{1}{5}x^5 \ln x - \ldots$	3
			•² differentiate to find "$\int u'v\,dx$" [3]	•² $\ldots \frac{1}{5}\int x^5 \times \frac{1}{x}dx$	
			•³ complete integration [4]	•³ $\frac{1}{5}x^5 \ln x - \frac{x^5}{25} + c$	

Notes:

1. Where a candidate differentiates or integrates both x^4 and $\ln x$, award 0/3.
2. Award 0/3 for candidates who differentiate x^4 and incorrectly integrate $\ln x$. See COR A if $\ln x$ is integrated correctly.
3. Do not withhold •² for the omission of dx.
4. Do not withhold •³ for the omission of the constant of integration.

Commonly Observed Responses:

COR A

•¹ $x^4(x\ln x - x) - \ldots$

•² $\ldots 4\int x^3(x\ln x - x)dx$

•³ $\frac{1}{5}x^5 \ln x - \frac{x^5}{25} + c$

COR B

Award •¹:

	Differentiate	Integrate
$+$	$\ln x$	x^4
$-$	$\frac{1}{x}$	$\frac{1}{5}x^5$

Award •² and •³ as per main method.

Candidates may have different headings, including u and v' for Differentiate and Integrate respectively.

Question			Generic scheme	Illustrative scheme	Max mark
5.			•1 construct auxiliary equation 1	•1 $m^2-4m-5=0$	9
			•2 find complementary function 2,3	•2 $y=Ae^{5x}+Be^{-x}$	
			•3 state particular integral and obtain first and second derivatives of particular integral 5	•3 $y=Cx^2+Dx+E$ $\frac{dy}{dx}=2Cx+D$ $\frac{d^2y}{dx^2}=2C$	
			•4 substitute into LHS of differential equation 4	•4 $2C-4\left(2Cx+D\right)-5\left(Cx^2+Dx+E\right)$	
			•5 obtain constants	•5 $C=-2, D=1$ and $E=3$	
			•6 state general solution 2,3,6,7	•6 $y=Ae^{5x}+Be^{-x}-2x^2+x+3$ stated or implied by •9	
			•7 differentiate general solution	•7 $\frac{dy}{dx}=5Ae^{5x}-Be^{-x}-4x+1$	
			•8 form simultaneous equations	•8 $A+B=-1$ $5A-B=13$	
			•9 state particular solution 2,3	•9 $y=2e^{5x}-3e^{-x}-2x^2+x+3$	

Notes:

1. •1 is not available where '$=0$' has been omitted.
2. •2 may still be awarded if the complementary function appears only as part of a general solution or the particular solution.
3. Do not withhold •2 for the omission of '$y=\ldots$' provided it appears as part of a general solution or at •6 or •9.
4. For the award of •4 a candidate must substitute an expression with variable coefficients.
5. Where a candidate does not introduce a particular integral only •1, •2, •7 and •8 are available.
6. Where a candidate includes as part of their general solution

a. $10x^2+11x-23$,
b. any expression containing constants other than those from the complementary function which have not been evaluated or
c. an incorrect expression which has not previously been identified as a particular integral,

•6 is unavailable but •7 may still be available.

7. Where a candidate introduces a particular integral after determining values for A and B, leading to $y=\frac{8}{3}e^{5x}-\frac{2}{3}e^{-x}-2x^2+x+3$, •6 is unavailable.

Question			Generic scheme	Illustrative scheme	Max mark
6.	(a)		•¹ find modulus or argument [1]	•¹ $r=2$ or $\theta=\frac{\pi}{3}$, (stated or implied at •²)	2
			•² complete polar form [1]	•² $2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$	

Notes

1. Where candidates work in degrees (60°) the degree symbol must appear at least once in part (a) or (b) for •² to be awarded.

Commonly Observed Responses:

Question			Generic scheme	Illustrative scheme	Max mark
	(b)		•³ apply de Moivre's Theorem	•³ $2^3\left(\cos\frac{3\pi}{3}+i\sin\frac{3\pi}{3}\right)$	2
			•⁴ demonstrate that imaginary part is zero [1,2,3,4]	•⁴ eg $z^3=-8$	

Notes:

1. Accept "$\sin\pi=0$, therefore z^3 is real", with no evaluation of the real part, for •⁴.
2. Where a candidate loses •³ as a result of an error, •⁴ is still available provided a consistent real number is produced.
3. Where an incorrect result is produced in part (a), •⁴ is available only if a consistent real number is produced.
4. Where a candidate chooses to evaluate z^3, the value must be consistent with the expression at •³.

Commonly Observed Responses:

COR A

Use of binomial theorem:

$1+3\sqrt{3}i+9i^2+3\sqrt{3}i^3$ award •³

COR B

Multiplying out one pair of brackets and resolving i^2:

$\left(1+\sqrt{3}i\right)\left(1+2\sqrt{3}i-3\right)$ award •³

COR C

Multiplying out all three brackets and resolving i^2 without attempting simplification of one pair:

$1+\sqrt{3}i+2\sqrt{3}i+\left(-6\right)+\left(-3\right)\ldots$ award •³

Question			Generic scheme	Illustrative scheme	Max mark
7.	(a)		•[1] substitute formulae •[2] simplify	•[1] $\frac{n(n+1)(2n+1)}{6}+3\left(\frac{n(n+1)}{2}\right)$ •[2] $\frac{1}{3}n(n+1)(n+5)$	**2**
Notes:					
Commonly Observed Responses:					
	(b)		•[3] substitute 20 and evidence of subtraction •[4] substitute 10 and evaluate	•[3] $\frac{1}{3}(20)(20+1)(20+5)-\ldots$ •[4] 2950	**2**
Notes:					
Commonly Observed Responses:					

Question			Generic scheme	Illustrative scheme	Max mark
8.	(a)		•[1] provide counterexample and demonstrate [1,2,3]	•[1] $a=-2, b=1$ 4 is not less than 1 or $4>1$	1

Notes:

1. The values of a and b must be explicitly stated and the squares evaluated.
2. Disregard any statement following a suitable demonstration.
3. Where a candidate chooses eg $a=-2$ and $b=2$, then it is sufficient to observe that $4=4$.

Commonly Observed Responses:

Question			Generic scheme	Illustrative scheme	Max mark
	(b)		•[2] appropriate form for n [1,2] •[3] factorise and communicate [3,4,5]	•[2] eg $2k+1$, $k\in\mathbb{Z}$ •[3] $4\left(k^2+k\right)$ and eg which is divisible by 4	2

Notes:

1. At •[2], accept eg "k is an integer" but do not accept eg $k\in\mathbb{N}$, $k\in\mathbb{Z}^{+}$.
2. Expression for n must be of the form $2k+m$, where m is an odd integer.
3. At •[3], accept $4k\left(k+1\right)$ for the factorisation.
4. Award •[3] if a candidate does not factorise but states that each term or coefficient (or equivalent) is divisible by 4.
5. Acceptable communication for •[3] includes "therefore true", "$\Rightarrow$ true", "so statement is true", "as required". Simply writing "true" after factorised expression is insufficient.

Commonly Observed Responses:

<table>
<tr><th colspan="3">Question</th><th>Generic scheme</th><th>Illustrative scheme</th><th>Max mark</th></tr>
<tr><td>9.</td><td>(a)</td><td></td><td>•¹ state A [1]</td><td>•¹ $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$</td><td>1</td></tr>
<tr><td colspan="6">Notes:
1. Accept $\begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix}$.</td></tr>
<tr><td colspan="6">Commonly Observed Responses:</td></tr>
<tr><td></td><td>(b)</td><td>(i)</td><td>•² find AB [1,2]</td><td>•² $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$</td><td>1</td></tr>
<tr><td colspan="6">Notes:
1. Where a candidate produces the identity matrix at (a), •² is not available.
2. All entries must be evaluated for the award of •².</td></tr>
<tr><td colspan="6">Commonly Observed Responses:</td></tr>
</table>

Question			Generic scheme	Illustrative scheme	Max mark
9.	(b)	(ii)	$\bullet^3$ find α [1,2,3,4]	$\bullet^3$ $\dfrac{5\pi}{3}$	1

Notes:

1. Do not accept an answer in degrees.
2. If the matrix in part (b)(i) is incorrect, $\bullet^3$ is available on follow-through only if the matrix is clearly identified as AB, or is the result of matrix multiplication.
3. Accept $\alpha = \dfrac{5\pi}{3} + 2k\pi,\ k \in \mathbb{Z}$, eg $\boldsymbol{\alpha} = -\dfrac{\boldsymbol{\pi}}{\mathbf{3}}$.
4. Where a candidate produces an incorrect matrix at (b)(i) but a correct angle at (ii), $\bullet^3$ is available only where the value is supported by a correctly applied valid strategy, eg adding the angles associated with matrices A and B.

Commonly Observed Responses:

Question			Generic scheme	Illustrative scheme	Max mark
	(c)		$\bullet^4$ find least value of n [1,2]	$\bullet^4$ 6	1

Notes:

1. Where a candidate produces an incorrect angle in (b)(ii), $\bullet^4$ is available only if n is greater than or equal to 3.
2. Where a candidate produces a value of n by calculating successive powers of a matrix from (b)(i), $\bullet^4$ is available only if n is greater than or equal to 3.

Commonly Observed Responses:

[END OF MARKING INSTRUCTIONS]

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2023 Mathematics

Advanced Higher - Paper 2

Finalised Marking Instructions

Marking Instructions for each question

Question			Generic scheme	Illustrative scheme	Max mark
1.			•¹ start differentiation	•¹ $\dfrac{2}{\sqrt{1-(3x)^2}}$	2
			•² apply chain rule [1]	•² $\dfrac{6}{\sqrt{1-(3x)^2}}$	

Notes:

1. At •² accept $\dfrac{6}{\sqrt{1-9x^2}}$ but do not accept $\dfrac{6}{\sqrt{1-3x^2}}$.

Commonly Observed Responses:

Question			Generic scheme	Illustrative scheme	Max mark
2.			•¹ evidence of recognising $\int \dfrac{f'(x)}{f(x)}dx$ [1,2]	•¹ $k\ln\lvert x^3+10\rvert,\ k\in\mathbb{R}$	2
			•² determine coefficient of $\ln\lvert x^3+10\rvert$ [1,2,3]	•² $\dfrac{1}{3}\ln\lvert x^3+10\rvert+c$	

Notes:

1. •¹ and •² are available only for an expression of the form $k\ln\lvert x^3+10\rvert,\ k\in\mathbb{R}$, with no further x terms.
2 Do not withhold •[1,2] for the omission of modulus signs.
3. Do not withhold •² for the omission of the constant of integration.

Commonly Observed Response:

Integration by Substitution:

$\dfrac{1}{3}\int\dfrac{du}{u}$ award •¹

$\dfrac{1}{3}\ln\lvert x^3+10\rvert+c$ award •²

Question			Generic scheme	Illustrative scheme	Max mark
3.	(a)		•¹ begin to find determinant •² simplify [1]	•¹ eg $2\begin{vmatrix}-1 & 0\\ 0 & -2\end{vmatrix} - 2x\begin{vmatrix}x & 0\\ 1 & -2\end{vmatrix} + 4\begin{vmatrix}x & -1\\ 1 & 0\end{vmatrix}$ •² $4x^2+8$	**2**

Notes:

1. Do not award •² where a candidate incorrectly simplifies $4x^2+8$.

Commonly Observed Response:

Rule of Sarrus

$$\begin{array}{ccccc} 2 & 2x & 4 & 2 & 2x \\ x & -1 & 0 & x & -1 \\ 1 & 0 & -2 & 1 & 0 \end{array}$$

award •¹

$4x^2+8$ award •²

Question			Generic scheme	Illustrative scheme	Max mark
3.	(b)		•³ state conclusion [1,2,3,4]	•³ eg $4x^2+8\neq 0$ A^{-1} always exists	**1**

Notes:

1. Accept $4x^2+8>0$.
2. Do not withhold •³ for omission of "always".
3. Accept 'yes' as conclusion after $4x^2+8\neq 0$.
4. Where the answer contains incorrect information (before, between or after correct information), •³ is not available.

Commonly Observed Responses:

COR A – Candidate produces a quadratic expression which would have a negative discriminant

$\mathbf{4x^2+2x+8\neq 0}$, so A^{-1} (always) exists — award •³

COR B

$4x^2\neq 0$

$x\neq 0$, so A^{-1} does not always exist — award •³

COR C

$\mathbf{4x^2+8=0}$

$\mathbf{x=(\pm)\sqrt{-2}}$,

so A^{-1} (always) exists — award •³

COR D

$\mathbf{4x^2+8=0}$

$\mathbf{x=(\pm)\sqrt{-2}}$,

so A^{-1} (always) exists except when $\mathbf{x=(\pm)\sqrt{-2}}$ — do not award •³

COR E

$\mathbf{4x^2+8=0}$

$\mathbf{x=(\pm)\sqrt{2}i}$, so A^{-1} does not exist for $x=(\pm)\sqrt{2}i$ — do not award •³

Question			Generic scheme	Illustrative scheme	Max mark
4.			•¹ begin differentiation of product term, with one term correct •² complete differentiation of product term •³ complete differentiation and calculate gradient	•¹ $2xy^2+\ldots$ or $\ldots+2x^2y\frac{dy}{dx}$ •² $2xy^2+2x^2y\frac{dy}{dx}$ •³ $2xy^2+2x^2y\frac{dy}{dx}-2\frac{dy}{dx}=3\cos 3x$ leading to $-\frac{3}{2}$	**3**

Notes:

1. Where the differentiation of the product term produces one term only, •¹ and •² are not available.
2. At •³, accept $\frac{3}{-2}$.

Commonly Observed Responses:

Question			Generic scheme	Illustrative scheme	Max mark
5.	(a)		•[1] state general term [1,3]	•[1] $\begin{pmatrix} 8 \\ r \end{pmatrix}(3x)^{8-r}\left(\dfrac{-2}{x^2}\right)^r$	3
			•[2] simplify powers of x or coefficients [3]	•[2] $3^{8-r}(-2)^r$ or x^{8-3r}	
			•[3] state simplified general term [2,3,4]	•[3] $\begin{pmatrix} 8 \\ r \end{pmatrix}3^{8-r}(-2)^r x^{8-3r}$	

Notes

1. Candidates may also proceed from $\begin{pmatrix} 8 \\ r \end{pmatrix}(3x)^r\left(\dfrac{-2}{x^2}\right)^{8-r}$.
2. Award full marks if the expression at •[3] appears without working.
3. Where a candidate writes out a full expansion, •[1], •[2] and •[3] are not available, unless the general term is identifiable in (b).
4. Where a candidate in (a) produces an incorrect further simplification subsequent to the correct answer (eg $3^{8-r}(-2)^r$ becomes $(-6)^8$), •[3] is not available.

Commonly Observed Responses:

COR A

General term has not been isolated

$$\sum_{r=0}^{8}\binom{8}{r}(3x)^{8-r}\left(\frac{-2}{x^2}\right)^r$$

$$=\sum_{r=0}^{8}\binom{8}{r}(3)^{8-r}(-2)^r x^{8-3r}$$

Do not award •[1]. Award •[2] and •[3].

COR B

General term has been isolated

$$\sum_{r=0}^{8}\binom{8}{r}(3x)^{8-r}\left(\frac{-2}{x^2}\right)^r$$

$$=\binom{8}{r}(3)^{8-r}(\ \ 2)^r x^{8-3r} \quad -$$

Disregard the incorrect use of the final equals sign. Award •[1], •[2] and •[3].

COR C

Binomial expression has been equated to the general term

$$\left(3x-\frac{2}{x^2}\right)^8=\binom{8}{r}(3x)^{8-r}\left(\frac{-2}{x^2}\right)^r$$

Disregard the incorrect use of the equals sign. Award •[1].

COR D

Negative sign omitted

$$\binom{8}{r}(3x)^{8-r}\left(\frac{2}{x^2}\right)^r$$

Do not award •[1], but •[2] and •[3] are still available.

COR E

Brackets omitted around -2

$$\binom{8}{r}(3)^{8-r}-2^r x^{8-3r}$$

Do not award •[3].

Question			Generic scheme	Illustrative scheme	Max mark
5.	(b)		•4 determine value of r [2] •5 find coefficient [1,2]	•4 3 •5 -108864	**2**

Notes:

1. Do not withhold •5 for an answer of $-\frac{108864}{x}$.
2. Where a candidate writes out a full expansion, •4 may be awarded only if the expansion is complete and correct at least as far as the required term (in either direction). The required term must clearly identified in the expansion for •5 to be awarded.

Commonly Observed Response:

Binomial expansion

$$6561x^8 - 34992x^5 + 81648x^2 - 108864x^{-1} + 90720x^{-4} - 48384x^{-7} + 16128x^{-10} - 3072x^{-13} + 256x^{-16}$$

Question			Generic scheme	Illustrative scheme	Max mark
6.	(a)		$\bullet^1$ obtain d [1]	$\bullet^1$ 19	1

Notes:

1. For the award of $\bullet^1$, 19 must be clearly identified as the gcd in (a) or implied by its use in (b).

Commonly Observed Responses:

Question			Generic scheme	Illustrative scheme	Max mark
	(b)		$\bullet^2$ express gcd in terms of 304 and 399 $\bullet^3$ find values of a and b [1]	$\bullet^2$ $19 = 304 - 3\times 95$ $= 304 - 3\times(399 - 1\times 304)$ $\bullet^3$ $a = 4, b = -7$	2

Notes:

1. At $\bullet^3$, where candidates do not explicitly communicate a and b:
   - accept $19 = 4\times 703 + (-7)\times 399$ and $19 = 703\times 4 + 399\times(-7)$
   - do not accept $19 = 4\times 703 - 7\times 399$ or $19 = 4\times 703 - 399\times 7$

Commonly Observed Responses:

Question			Generic scheme	Illustrative scheme	Max mark
	(c)		$\bullet^4$ find values of p and q [1,2]	$\bullet^4$ $p = 16, q = -28$	1

Notes:

1. Do not accept $a = 16, b = -28$.
2. At $\bullet^4$, where candidates do not explicitly communicate p and q:
   - accept $76 = 16\times 703 + (-28)\times 399$ and $76 = 703\times 16 + 399\times(-28)$
   - do not accept $76 = 16\times 703 - 28\times 399$ $76 = 16\times 703 - 399\times 28$ unless $\bullet^3$ has already been withheld on the same grounds.

Commonly Observed Responses:

Question			Generic scheme	Illustrative scheme	Max mark
7.	(a)		•[1] find integrating factor [1] •[2] write as integral equation [2,3] •[3] integrate right-hand side [4,5] •[4] find particular solution [5,6]	•[1] e^{-2x} •[2] $e^{-2x}y=6\int e^{5x}e^{-2x}\,dx$ •[3] $6\times\frac{1}{3}e^{3x}+c$ •[4] $y=2e^{5x}-3e^{2x}$	**4**

Notes:

1. For any attempt to separate variables, award 0/4.
2. Do not withhold •[2] for the omission of 'dx' on the right-hand side.
3. Where a candidate writes $\int \frac{d}{dx}...$ on the left-hand side do not withhold •[2] provided the candidate later produces evidence that they have integrated with respect to x.
4. Where a candidate integrates $6e^{5x}$, •[3] is not available.
5. For a candidate who omits the constant of integration,•[3] may be available but •[4] is not.
6. Accept $y=\frac{2e^{3x}-3}{e^{-2x}}$ at •[4].

Commonly Observed Response:

Candidate treats equation as linear differential equation:

•[1] write auxiliary equation and obtain complementary function: $m-2=0$ and $y=Ae^{2x}$

•[2] particular integral and derivative: $y=Be^{5x}$ and $\frac{dy}{dx}=5Be^{5x}$

•[3] substitute PI into differential equation and determine B: $5Be^{5x}-2Be^{5x}=6e^{5x}$ and $B=2$

•[4] find particular solution: $y=2e^{5x}-3e^{2x}$

Question			Generic scheme	Illustrative scheme	Max mark
	(b)		•[5] find third derivative [1,2] •[6] find k [2,3]	•[5] $\frac{d^3y}{dx^3}=250e^{5x}\quad 24e^{2x}$ •[6] k = 36	**2**

Notes:

1. Where a candidate recognises from the RHS that the e^{5x} term need not be considered, •[5] is available only where this is clearly stated.
2. Where a candidate adopts an approach using an auxiliary equation, •[5] is available for a general complementary function of the form $y=A+Bx+Ce^{5x}$ AND a particular integral of the form $y=De^{2x}$. For the award of •[6], candidates must derive the value of k from their coefficient of e^{2x} in (a).
3. Accept $36e^{2x}$ for the award of •[6].

Commonly Observed Responses:

Question			Generic scheme	Illustrative scheme	Max mark
8.	(a)	(i)	•1 find the common ratio	•1 3	1
Notes:					
Commonly Observed Responses:					
		(ii)	•2 find first term	•2 $\frac{1}{3}$	1
Notes:					
Commonly Observed Responses:					
	(b)		•3 find S_n and S_{2n} [1] •4 obtain expression [2]	•3 $S_n = \frac{\frac{1}{3}\left(1-3^n\right)}{1-3}$ and $S_{2n} = \frac{\frac{1}{3}\left(1-3^{2n}\right)}{1-3}$ •4 $\frac{\left(1-3^n\right)\left(1+3^n\right)}{1-3^n}$ leading to $1+3^n$	2

Notes:

1. For the award of •3, where a candidate produces a fractional common ratio in (a), brackets must be used when substituting, unless they separately consider numerator and denominator.
2. For the award of •4, there must be evidence that the candidate has completed a difference of two squares, or multiplied numerator and denominator by $1+3^n$.

Commonly Observed Response:

For candidates who deal with general expressions:

$\frac{1-r^{2n}}{1-r^n}$ or $\frac{r^{2n}-1}{r^n-1}$ award •3

Question			Generic scheme	Illustrative scheme	Max mark
9.			•1 evidence of valid method 1 •2 express in base nine 2,3	$572 = 9 \times 63 + 5$ •1 $63 = 9 \times 7 + 0$ $7 = 9 \times 0 + 7$ •2 705_9	2

Notes:

1. For the award of •1 , the final expression must contain a zero quotient.
2. For the award of •2, at least three digits must be produced from a valid method.
3. At •2, disregard omission of base 9.

Commonly Observed Responses:

COR A

$572 \div 9 = 63 \text{ r } 5$

$63 \div 9 = 7 \text{ r } 0$

$7 \div 9 = 0 \text{ r } 7$

leading to 705 award •1 and •2

COR B

$572 = 9 \times 63 + 5$

$63 = 9 \times 7 + 0$

leading to 705 award •1 and •2

COR C

$572 = 9 \times 63 + 5$

$63 = 9 \times 7 + 0$

leading to 507 award •1 but not •2

COR D

9^2	9^1	9^0
81	9	1
7	0	5

Award •1 for all entries in row 2 and the '7' in row 3.

Question			Generic scheme	Illustrative scheme	Max mark
10.			•1 take logarithms on both sides and apply rule 1 •2 differentiate $\ln y$ •3 evidence of product rule with one term correct 2,3 •4 complete differentiation 2,3 •5 write $\frac{dy}{dx}$ in terms of x 1,3,4	•1 $\ln y = 5x^2 \ln x$ •2 $\frac{1}{y}\frac{dy}{dx}$ •3 $10x\ln x + \ldots$ or $\ldots + 5x^2 \cdot \frac{1}{x}$ •4 $10x\ln x + 5x^2 \cdot \frac{1}{x}$ •5 $\frac{dy}{dx} = x^{5x^2}\left(10x\ln x + 5x\right)$	5

Notes:

1. Accept 'log' as an alternative to '$\ln$' provided candidate does not indicate a base other than e. For candidates who do use a base other than e, only •1 and •5 are available.
2. For candidates who do not attempt to use the product rule, •3 and •4 are not available.
3. Accept '$5x$' instead of '$5x^2 \cdot \frac{1}{x}$' for •3 and •4 . However, do not accept '$5x^2 \cdot \frac{1}{x}$' for •5 .
4. •5 is not available for candidates who subsequently produce an incorrect statement - eg $x^{5x^2}\left(10x\ln x + 5x\right)$ becomes $x^{6x^2}\left(10\ln x + 5\right)$.

Commonly Observed Responses:

COR A

For candidates who write $y = e^{\ln x^{5x^2}}$, marks may be awarded as follows.

•1 write in the form $y = e^{\ln x^{5x^2}}$.

•2,3 apply chain rule $\frac{dy}{dx} = e^{5x^2 \ln x} \cdot \frac{d}{dx}\left(5x^2 \ln x\right)$

•4 use product rule with one term correct $10x\ln x + \ldots$ or $\ldots + 5x^2 \cdot \frac{1}{x}$

•5 complete differentiation $\frac{dy}{dx} = x^{5x^2}\left(10x\ln x + 5x\right)$ or $\frac{dy}{dx} = e^{5x^2 \ln x}\left(10x\ln x + 5x\right)$

COR B

For candidates who write $y = e^{5x^2} \ln x$ do not award •1, •2 or •5. However, •3 and •4 are still available for a correct application of the product rule.

Question			Generic scheme	Illustrative scheme	Max mark
11.	(a)		•1 determine the relationship between r and h	•1 $r=\frac{90h}{150}$ (or equivalent) leading to $V=\frac{3\pi h^3}{25}$	1

Notes:

Commonly Observed Responses:

Question			Generic scheme	Illustrative scheme	Max mark
	(b)		•2 find $\frac{dV}{dh}$ 2	•2 $\frac{dV}{dh}=\frac{9\pi h^2}{25}$	5
			•3 form relationship 3	•3 eg $\frac{dV}{dt}=\frac{dV}{dh}\frac{dh}{dt}$ stated or implied at •5	
			•4 interpret rate of change of V in cm^3s^{-1}	•4 $\frac{dV}{dt}=10000$	
			•5 form expression for $\frac{dh}{dt}$ in terms of h 5	•5 $\frac{dh}{dt}=\frac{25\times10000}{9\pi h^2}$	
			•6 evaluate $\frac{dh}{dt}$ 1,4,6	•6 $\frac{16}{9\pi}\text{cms}^{-1}$	

Notes:

1. Where alternative notation is used for V, h and t, •6 is available only if the variables are defined and used consistently.
2. If a derivative is equated to the original expression, •2 is not available.
3. At •3 accept any correct form of the chain rule which involves both $\frac{dh}{dt}$ and $\frac{dV}{dt}$.
4. Accept an answer rounded to at least 2 significant figures $\left(0.57\ \text{cms}^{-1}\right)$.
5. For the award of •5 the expression need not be simplified.
6. Units are required for •6.

Commonly Observed Response:

Where a candidate does not convert litres to cm³.

$\frac{dV}{dt}=10$ do not award •4

•5 $\frac{dh}{dt}=\frac{25\times10}{9\pi h^2}$

•6 $\frac{2}{1125\pi}\text{cms}^{-1}$ or $5.7\times10^{-4}\text{cms}^{-1}$

Question			Generic scheme	Illustrative scheme	Max mark
12.			●¹ show true for $n = 1$ [1]	●¹ $(\text{LHS} =) 2^{1-1}\ \ 1 \times 1$ $(\text{RHS} =) 2^1(1\ \ 1) - 1 + 1$ so result is true when $n = 1$	**5**
			●² assume statement true for $n = k$ AND consider whether statement true for $n = k + 1$ [2,5]	●² suitable statement AND $\sum_{r=1}^{k} 2^{r-1} r = 2^k(k-1)+1$ AND $\sum_{r=1}^{k+1} 2^{r-1} r = \cdots$	
			●³ state sum to $k + 1$ terms using inductive hypothesis [3]	●³ $2^k(k-1)+1+(k+1)2^{k+1-1}$ or $2^k(k-1)+1+(k+1)2^k$	
			●⁴ take out common factor of 2^k and simplify [4]	●⁴ $2^k.2k+1$	
			●⁵ express sum explicitly in terms of $(k + 1)$ or achieve stated aim/goal AND communicate [5,6,7]	●⁵ $2^{(k+1)}(k+1-1)+1$ **AND** If true for $n = k$ then true for $n = k + 1$. Also true for $n = 1$ therefore, by induction, true for all positive integers n	

Question		Generic scheme	Illustrative scheme	Max mark
12.	(continued)			

Notes:

1. "RHS = 1, LHS = 1" and/or "True for $n = 1$" are insufficient for the award of $\bullet^1$. Where a candidate does not independently evaluate LHS and RHS, $\bullet^1$ may still be awarded.

2. For $\bullet^2$, sufficient phrases for $n = k$ contain:
   - 'If true for...'; 'Suppose true for…'; 'Assume true for…'.

   For $\bullet^2$ insufficient phrases for $n = k$ contain:
   - 'Consider $n = k$', 'assume $n = k$', 'assume $n = k$ is true' and 'True for $n = k$'.

   A sufficient phrase for the award of $\bullet^2$ may appear at $\bullet^5$.

   For $\bullet^2$, accept:

   assume true for $n = k$ AND $\sum_{r=1}^{k} 2^{r-1} r = 2^k (k-1) + 1$ AND "Aim/goal: $\sum_{r=1}^{k+1} 2^{r-1} r = 2^{k+1}(k+1-1)+1$"

   For $\bullet^2$ unacceptable phrases for $n = k + 1$ contain:
   - "Consider true for $n = k + 1$", "true for $n = k + 1$",
   - "$\sum_{r=1}^{k+1} 2^{r-1} r = 2^{k+1}(k+1-1)+1$" (with no reference to aim/goal and no further processing).

3. Since $\bullet^3$ may be awarded directly after $\bullet^2$, exercise leniency in relation to intermediate notation where this has been provided, eg the handling of sigma notation.

4. $\bullet^4$ is unavailable to candidates who arrive at $2^k.2k+1$ without algebraic justification.

5. $\bullet^5$ is unavailable to candidates who have not been awarded $\bullet^4$.

6. Full marks are available to candidates who state an aim/goal earlier in the proof and who subsequently achieve the stated aim/goal, provided $2^{k+1}(k+1-1)+1$ appears at some point.

7. Following the required algebra and statement of the inductive hypothesis, the minimal acceptable response for $\bullet^5$ is: "Then true for $n = k + 1$, but since true for $n = 1$, then true for all n" or equivalent.

Commonly Observed Responses:

Question			Generic scheme	Illustrative scheme	Max mark
13.			•[1] write as integral equation [1]	•[1] $\int \frac{1}{P}dP = \int \frac{1\cdot4}{m-220}dm$	6
			•[2] integrate P expression	•[2] $\ln P$	
			•[3] integrate m expression [2]	•[3] $1\cdot4\ln(m-220)+c$	
			•[4] substitute values following integration [2]	•[4] $\ln 1079 = 1\cdot4\ln(807 - 220) + c$	
			•[5] evaluate constant of integration [2,4]	•[5] $-1\cdot94$	
			•[6] write expression in terms of m [2,3,4]	•[6] $P = 0\cdot14(m-220)^{1\cdot4}$	

Notes:

1. Do not award •[1] where $\int \ldots dP$ and $\int \ldots dm$ do not appear.
2. For candidates who omit the constant of integration, •[3] may be awarded but •[4], •[5] and •[6] are unavailable.
3. For •[6] accept $P = e^{1\cdot4\ln(m-220)-1\cdot94}$ or equivalent.
4. Disregard numerical errors due to truncation or premature rounding.

Question	Generic scheme	Illustrative scheme	Max mark

13. (continued)

Commonly Observed Responses:

Alternative Correct Solutions

COR A (rearranging before substitution)

•4 $P = e^{1.4\ln(m-220)+c}$

•5 $1079 = e^{1.4\ln(807-220)}e^{c}$

•6 $P = 0{\cdot}14(m-220)^{1.4}$

COR B

•1 $\displaystyle\int \frac{1}{1{\cdot}4P}dP = \int \frac{1}{m-220}dm$

•2 $\dfrac{1}{1.4}\ln P$

•3 $\ln(m-220)+c$

•4 $\dfrac{1}{1{\cdot}4}\ln 1079 = \ln(807-220)+c$

•5 $-1{\cdot}39$

•6 $P = 0{\cdot}14(m-220)^{1.4}$

COR C

•1 $\displaystyle\int \frac{1}{1{\cdot}4P}dP = \int \frac{1}{m-220}dm$

•2 $\dfrac{1}{1.4}\ln 1.4P$

•3 $\ln(m-220)+c$

•4 $\dfrac{1}{1{\cdot}4}\ln(1.4\times1079) = \ln(807-220)+c$

•5 $-1{\cdot}15$

•6 $P = 0{\cdot}14(m-220)^{1.4}$

Incorrect Integration of LHS

COR D

•1 $\displaystyle\int \frac{1}{1{\cdot}4P}dP = \int \frac{1}{m-220}dm$

$1.4\ln P$ — do not award •2

•3 $\ln(m-220)+c$

•4 $1{\cdot}4\ln 1079 = \ln(807-220)+c$

•5 3.40

•6 $P = 11.3(m-220)^{0.71}$

COR E

•1 $\displaystyle\int \frac{1}{1{\cdot}4P}dP = \int \frac{1}{m-220}dm$

$1.4\ln 1.4P$ — do not award •2

•3 $\ln(m-220)+c$

•4 $1{\cdot}4\ln(1.4\times1079) = \ln(807-220)+c$

•5 3.87

•6 $P = 11.3(m-220)^{0.71}$

COR F

•1 $\displaystyle\int \frac{1}{1{\cdot}4P}dP = \int \frac{1}{m-220}dm$

$\ln 1.4P$ — do not award •2

•3 $\ln(m-220)+c$

•4 $\ln(1.4\times1079) = \ln(807-220)+c$

•5 0.945

$P = 1.84(m-220)$ — do not award •6 (eased)

Question	Generic scheme	Illustrative scheme	Max mark
13. (continued)			
		<u>Incorrect Integration of LHS</u> **COR G** •[1] $\int \frac{1}{1 \cdot 4P} dP = \int \frac{1}{m-220} dm$ $\ln 1.4P$ — do not award •[2] •[3] $\ln(m-220)+c$ $P = \frac{(m-220)e^c}{1.4}$ — do not award •[4] (eased) •[5] $1079 = \frac{(807-220)e^c}{1.4}$ •[6] $P = 1.84(m-220)$	

Question			Generic scheme	Illustrative scheme	Max mark
14.			•1 substitute, expand and apply $i^2 = -1$ [1]	•1 $a^2 - b^2 + 2abi$	**4**
			•2 equate real and imaginary parts	•2 $a^2 - b^2 = 8$ and $2ab=6$	
			•3 substitute for b or a [2]	•3 eg $a^2 - \frac{9}{a^2} = 8$	
			•4 rearrange into quartic in standard form and solve [2,3,4]	•4 $a^4 - 8a^2 - 9 = 0$ and $a = 3, b = 1$	

Notes:

1. The imaginary part need not be simplified for the award of •1.
2. For the award of •3 and •4, there must be suitable algebraic justification. Answers obtained by trial and error are not acceptable.
3. For the award of •4, a and b must both be positive.
4. Provided appropriate algebraic justification is present, •4 may be awarded for $w = 3 + i$.

Commonly Observed Responses:

Candidates who use de Moivre's theorem:

•1 determine modulus or argument	$\lvert w^2 \rvert = 10$ or $\theta = 36.9°$ (0.64 radians)
•2 express in polar form	$10(\cos 36.9° + i\sin 36.9°)$ or $10(\cos 0.64 + i\sin 0.64)$
•3 apply de Moivre's theorem	$\sqrt{10}\left(\cos\frac{36.9°}{2} + i\sin\frac{36.9°}{2}\right)$ or $\sqrt{10}\left(\cos\frac{0.64}{2} + i\sin\frac{0.64}{2}\right)$
(Do not award •3 if the argument of w^2 is 0 or of the form $2k\pi$ or $360k°$, $k \in \mathbb{Z}$.)	
•4 determine a and b	$a = 3, b = 1$ or $a = 3, b = 0.99$

Question			Generic scheme	Illustrative scheme	Max mark
15.	(a)		•1 evidence of use of quotient rule with denominator and one term of numerator correct [1]	•1 $\dfrac{1+(x+1)^4 \ldots}{\left(1+(x+1)^4\right)^2}$ **OR** $\dfrac{\ldots-(x+1)4(x+1)^3}{\left(1+(x+1)^4\right)^2}$	**3**
			•2 complete differentiation	•2 $\dfrac{1+(x+1)^4-(x+1)4(x+1)^3}{\left(1+(x+1)^4\right)^2}$	
			•3 find required terms	•3 $1+\frac{1}{2}x-\frac{1}{4}x^2$	

Notes:

1. Where a candidate arrives at $1+\frac{1}{2}x-\frac{1}{4}x^2$ without algebraic differentiation, award •3 only.

Commonly Observed Responses:

COR A

Alternative Method (Product Rule)

•1 $\left(1+(x+1)^4\right)\ldots$

•2 $\ldots+(x+1)(-1)\left(1+(x+1)^4\right)^{-2}4(x+1)^3$

COR B

Candidates who use logarithmic differentiation

•1 $\ln\left(f'(x)\right)=\ln(x+1)-\ln\left(1+(x+1)^4\right)$

•2 $\dfrac{1}{f'(x)}f''(x)=\dfrac{1}{x+1}-\dfrac{4(x+1)^3}{1+(x+1)^4}$ (candidates may be less precise on LHS)

Question			Generic scheme	Illustrative scheme	Max mark
15.	(b)		•4 find $\frac{du}{dx}$ 1 •5 rewrite integral in terms of u 1 •6 integrate and substitute for u 2	•4 $\frac{du}{dx} = 2(x+1)$ •5 $\frac{1}{2}\int \frac{du}{1+u^2}$ •6 $\frac{1}{2}\tan^{-1}(x+1)^2 + c$	**3**

Notes:

1. $\frac{du}{dx}$ may be implied at •5.
2. Do not withhold •6 for the omission of the constant of integration.

Commonly Observed Responses:

Question			Generic scheme	Illustrative scheme	Max mark
	(c)		•7 interpret Maclaurin expansion 1 •8 obtain $f(x)$ 2	•7 $f(0) = 1$ •8 $\frac{1}{2}\tan^{-1}(x+1)^2 + 1 - \frac{\pi}{8}$	**2**

Notes:

1. At •7 accept $\frac{1}{2}\tan^{-1}(0+1)^2 + c = 1$.
2. Award •8 for $c = 1 - \frac{\pi}{8}$ provided candidate has previously written $\frac{1}{2}\tan^{-1}(x+1)^2 + c$.

Commonly Observed Responses:

[END OF MARKING INSTRUCTIONS]